Portfolio Insurance: the Extreme Value Approach to the CPPI Method

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1. INTRODUCTION

Portfolio insurance is designed to give the investor the ability to limit downside risk while allowing some participation in upside markets. This return pattern has seemed attractive to many investors who have poured up to billions of dollars into various portfolio insurance products. There exist many methods of insurance portfolio: OBPI (Option Based Portfolio Insurance), CPPI (Constant Proportion Portfolio Insurance), Stop-loss... (see for example Poncet and Portait (1997), Bertrand and Prigent (2001) for a comparison of these methods). Here, we are interested in a widely used one: the CPPI introduced by Black and Jones (1987) for equity instruments and Perold (1986, 1988) for fixed-income instruments (see also Black and Rouhani (1987), Roman, Kopprash and Hakanoglu (1989), Black and Perold (1992)).

The CPPI method uses a simplified strategy to allocate assets dynamically over time. It requires that two assets are exchanged on the financial market: the riskless asset, B, with a constant interest rate r (= 5%) (usually Treasury bills or other liquid money market instruments) and the risky one, S (usually a market index or a basket of market indexes).

To illustrate the method, consider for example an investor with initial amount to invest V_0 (= 100). Assume she wants to recover a prespecified percentage α (= 95%) of her initial investment at a given date in the future, T (1 year). Note that the insured terminal value αV_0 (= 95) can't be greater than the initial value

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capitalized at the riskfree rate, $V_0 \cdot e^{rT}$ (= 105,13). Her portfolio manager starts by setting an initial floor $F_0 = \alpha \cdot V_0 \cdot e^{-rT}$ (= 90,37). To obtain a terminal portfolio value V_T greater than the insured amount αV_0, he keeps the portfolio value V_t above the floor $F_t = \alpha \cdot V_0 \cdot e^{-r(T-t)}$ at any time t in the period $[0, T]$. For this purpose, the amount e_t invested in the risky asset is a fixed proportion m of the excess C_t of the portfolio value over the floor. The constant m is usually called *the multiple*, e_t *the exposure* and C_t *the cushion.* Since $C_t = V_t - F_t$, this insurance method consists in keeping C_t positive at any time t in the period. The remaining funds are invested in the riskless asset B_t.

Both the floor and the multiple are functions of the investor's risk tolerance. The higher the multiple, the more the investor will participate in a sustained increase in stock prices. Nevertheless, the higher the multiple, the faster the portfolio will approach the floor when there is a sustained decrease in stock prices. As the cushion approaches zero, exposure approaches zero, too. Normally, this keeps portfolio value from falling below the floor. Nevertheless, during financial crises a very sharp drop in the market may occur before the manager has a chance to trade. This implies that m must not be too high (for example, if a fall of 10% occurs, m must not be greater than 10 in order to keep the cushion positive).

Advantages of this strategy over other approaches to portfolio insurance are its simplicity and its flexibility (see for example De Vitry and Moulin (1994), Black and Rouhani (1987) and Boulier and Sikorav (1992)). Initial cushion, floor and tolerance can be chosen according to the own investor's objective (see for example Poncet and Portait (1997) and Prigent (2001*b*)). Note that the multiple can be derived from the maximization of the expected utility of the investor. However in that case, the optimal multiple is a function of the risky asset as shown in El Karoui, Jeanblanc and Lacoste (2000).

Usually, banks don't directly bear market risks on the asset portfolios they manage for their customers. This is not necessarily true when we consider management of insured portfolios. In that case, banks can use, for example, stress testing since they may bear consequences of sudden large market decreases, depending on their method of management ([1]). For instance, in the case of the CPPI method, banks must, at least, provision the difference on their own capital if the value of the portfolio drops below the floor.

So, one crucial question for the bank which promotes such funds is: what exposure to the risky asset or, equivalently, what level of the multiple to accept? On one hand, as portfolio expectation return is increasing with respect to the multiple, customers want the multiple as high as possible. On the other hand, due to market imperfections ([2]), portfolio managers must impose an upper bound on the multiple.

First, if the portfolio manager anticipates that the maximal daily historical drop (e.g. – 20%) will happen during the period, he chooses $m \leq 5$ which leads to low return expectation. Alternatively, he may think that the maximal daily drop during the period he manages the portfolio will never be greater than a given value (e.g. – 10%). A straightforward implication is to choose m according to this new extreme value (e.g. $m \leq 10$). Another possibility is that, more accurately, he takes account of the occurrence probabilities of extreme events in the risky asset returns. Finally, he may adopt a quantile hedging strategy: choose the multiple as high as possible but so that the portfolio value will always be above the floor at a given probability level (typically 99%).

The answer to the previous questions has important practical implications in the management process implementation. It is addressed in this paper, which is organized as follows. Section 2 presents the model and basic properties related to extreme value theory. Section 3 examines the CPPI method and provides upper bounds on the multiple, in particular when using a quantile hedging approach. Extreme value theory allows us to approximate these bounds. Finally, empirical estimation on the S&P500 series illustrates our approach.

2. THE MODEL

2.1. The financial market

Changes in asset prices are supposed to occur at discrete times along a whole period $[0, T]$ (for example, thirty years which may be the period of all observations of financial data). Particular subperiods indexed by l, $[l\theta, (l+1)\theta]_{l \leq p}$ with $p\theta = T$ can be introduced: they may correspond to standard portfolio management periods (for example one year). Finally, for each subperiod l, consider the sequence of deterministic prices variations times $(t_k^l)_k$ in $[l\theta, (l+1)\theta]$ that for simplicity, for each l, we denote by $(t_k)_k$ (for example, daily variations). The variations of the stock price S between two times t_k and t_{k+1} are defined by:

$$\Delta S_{t_{k+1}} = S_{t_{k+1}} - S_{t_k}.$$

Since we search an upper bound on the multiple m (see Proposition 1 in the following), we have to focus on the left hand side of the probability distribution of $\frac{\Delta S_{t_{k+1}}}{S_{t_k}}$. Thus, we introduce the notation:

$$X_{k+1} = -\frac{\Delta S_{t_{k+1}}}{S_{t_k}} = \frac{S_{t_k} - S_{t_{k+1}}}{S_{t_k}}.$$

So X_k denotes the opposite of the relative jump of the risky asset at time t_k.

Due to the possible anticipation of the portfolio manager concerning these relative jumps during a given period (for example, X_k will never be greater than 10%), we introduce also the truncated jumps $X_k^{[a,b]}$ defined by:

$$X_k^{[a,b]} = X_k \, 1_{a \leq X_k \leq b}$$

where $1_{a \leq X_k \leq b}$ is equal to 1 if $a \leq X_k \leq b$ and equal to 0 otherwise.

In fact, when determinating an upper bound on the multiple m, we have only to consider positive values of X_k (see section 3 in the following). So, it is sufficient to take $a = 0$ and b equal to the maximal relative drop anticipated for the management period. Since the portfolio manager may take account of the occurence probabilities of some given values of the relative jumps (corresponding for example to maximal drops), we must introduce their arrival times which are random variables. Denote $(T_k^{[a,b]})_k$ the sequence of times at which X_k takes values in the interval $[a, b]$. The sequence $(T_k^{[a,b]}, X_k^{[a,b]})_k$ is called a marked point process (3).

2.2. Extreme Value Theory

We measure stock market price movements by the daily rates of returns. More precisely, we use the opposite of the arithmetical rate X_k which determine the conditions to impose on the mutiple m, within the context of the CPPI method. If these variables are statistically independent, and drawn from the same distribution, then the exact distribution of the maximum of $X_1, \ldots, X_n$ is equal to the power F_X^n of the common distribution of the X_k. But, in most cases, this distribution is not exactly known. Nevertheless, like for the well-known central limit theorem, there exists a normalization procedure to get non-degenerate distributions at the limit. This is the fundamental result of the Extreme Value Theory, which is in particular detailed in Leadbetter, Lindgren and Rootzen (1983) or in Embrechts, Klüppelberg and Mikosch (1997).

Recall the Fisher-Tippett theorem (1928) concerning the limit laws for maxima: Let $(X_k)_k$ be a sequence of iid random variables. If there exist norming constants $\mu_k, \psi_k > 0$ and some non-degenerate distribution function H such that

$$\frac{\max(X_1, \cdots, X_n) - \mu_n}{\psi_n} \Rightarrow H$$

where $\Rightarrow$ denotes convergence in distribution, then H belongs to one of the following three distribution functions:

$$\begin{aligned} &\text{Fréchet:} && \Phi_\alpha(x) = \exp(-x^{-\alpha}),\ x > 0,\ \alpha > 0 \\ &\text{Weibull:} && \Psi_\alpha(x) = \exp(-(-x)^\alpha,\ x < 0,\ \alpha < 0 \\ &\text{Gumbel:} && \Lambda(x) = \exp(-e^{(-x)}),\ x \in \mathbb{R}. \end{aligned}$$

As shown by Jenkinson and von Mises, these three kinds of distributions are particular cases of the generalized extreme distributions (GEV) defined by:

$$H_\xi(x) = \begin{cases} \exp(-(1+\xi x)^{\frac{-1}{\xi}}) & \text{if } \xi \neq 0, \text{ where } 1+\xi x > 0 \\ \exp(-\exp(-x)) & \text{if } \xi = 0\cdot \end{cases}$$

Notice that the standard extreme distributions can be recovered with:

$\xi = \alpha^{-1}$ for the Fréchet distribution,
$\xi = 0$ for the Gumbell distribution,
$\xi = -\alpha^{-1}$ for the Weibull distribution.

The parameter ξ is called the *tail index* and $\frac{1}{\xi}$ is called the *shape index*. The statistical problem is to find the correct distribution of extremes of returns from the data and, in particular, to estimate the norming constants μ_n, ψ_n and ξ. For this purpose, when using a maximum likelihood method, we need the GEV distribution of a general non-centered, non-reduced random variable, defined by:

$$H_{\xi,\mu,\psi}(x) = \begin{cases} \exp\left(-\left(1+\xi\left(\frac{x-\mu}{\psi}\right)\right)^{\frac{-1}{\xi}}\right) & \text{if } \xi \neq 0, \text{ where } 1+\xi\left(\frac{x-\mu}{\psi}\right) > 0 \\ \exp\left(-\exp\left(-\left(\frac{x-\mu}{\psi}\right)\right)\right) & \text{if } \xi = 0. \end{cases}$$

When examining the sequence $X_{k+1} = \frac{S_{t_k} - S_{t_{k+1}}}{S_{t_k}}$, it is obvious that its distribution has a right end limit equal to 1. So, the normalized maxima $\frac{\max(X_1, \cdots, X_n) - \mu_n}{\psi_n}$ converge either to the Gumbel distribution Λ or to a Weibull distribution Ψ_α.

Recall the characterizations of the result of Resnik (1987) for the maximum domain of attraction of Λ, based on generalizations of the Von Mises functions (see also for example Embrechts and alii (1997)):

The distribution F with right end point limit $x_F \leq \infty$ belongs to the maximum domain of attraction of Λ if and only if there exists some $z < x_F$ such that F has representation:

$$\overline{F}(x) = c(x)\exp\left(-\int_z^x \frac{g(t)}{\alpha(t)}\,dt\right),\ z < x < x_F$$

where c and g are functions satisfying $c(x) \to c > 0$, $g(x) \to 1$ as $x \uparrow x_F$ and $\alpha(x)$ is a positive, absolutely continuous function with respect to Lebesgue measure with density $\alpha'(x)$ having $\lim_{x\uparrow x_F} \alpha'(x) = 0$.

Denote $F^{\leftarrow}(q)$ the q quantile of the distribution F, more precisely:

$$F^{\leftarrow}(q) = \inf \{x \in \mathbb{R}, F(x) \geq q.$$

For F with the previous representation, we can choose:

$$\mu_n = F^{\leftarrow}\left(1 - \frac{1}{n}\right) \text{ and } \psi_n = \alpha(\mu_n).$$

A possible choice of the function $\alpha(\cdot)$ is:

$$\alpha(x) = \int_x^{x_F} \frac{\overline{F}(t)}{\overline{F}(x)}\, dt,\ x < x_F.$$

Besides, if we truncate the jumps X_k on the interval $[a, b]$ with $b < 1$ and examine the sequence $(X_k^{[a,b]})_k$ then we can check that there exist norming constants $\mu_k^{[a,b]}$ and $\psi_k^{[a,b]}$ such that:

$$\frac{\max(X_1^{[a,b]}, \ldots, X_n^{[a,b]}) - \mu_n^{[a,b]}}{\psi_n^{[a,b]}} \to \Lambda.$$

3. CPPI and Quantile Hedging

3.1. Upper Bounds on the Multiple

Recall that $X_k = -\dfrac{S_k - S_{k-1}}{S_{k-1}}$ and $M_n = \max(X_1, \ldots, X_n)$. Denote by V_k the value of the portfolio at time t_k. As explained in the introduction, the CPPI method is based on the following portfolio insurance condition: there exists a deterministic floor F_k such that at any time t_k, the value V_k must be above the floor. The total amount e_k invested on the underlying asset S is equal to mC_k where the cushion C_k represents the difference $V_k - F_k$ between the portfolio value and the floor. The multiple m is a non negative constant. The remaining amount $(V_k - e_k)$ is invested on the riskless asset with a deterministic rate r_k for the period $[t_{k-1}, t_k]$.

The higher the multiple m, the higher the amount invested on the risky asset. So a speculative investor would choose high values for m. Nevertheless, in that case, her portfolio is riskier and as shown in what follows, her guarantee may no longer hold. Indeed, we easily deduce that the portfolio value is solution of:

$$V_{k+1} = V_k - e_k X_{k+1} + (V_k - e_k)\, r_{k+1},$$

from which, we obtain the dynamics of the cushion:

$$C_{k+1} = C_k[1 - mX_{k+1} + (1 - m)\, r_{k+1}],$$

Since, for all times t_k, the cushion must be positive, we get finally the condition: for all $k \leq n$,

$$-mX_k + (1-m)\, r_k \geq -1.$$

In fact, since r_k is relatively small, the previous inequality yields to the following relation that gives an upper bound on the multiple:

Proposition 1:

$$\forall k \leq n,\ X_k \leq \frac{1}{m} \text{ or equivalently } M_n = \max(X_k)_{k \leq n} \leq \frac{1}{m}.$$

Since the right end point d of the common distribution F of the variables X_k is positive, we deduce that the insurance is perfect along any period $[0, T']$ *if and only if m is smaller than* $\frac{1}{d}$.

For example, if the maximal drop is equal to 20%, then $d = 0, 2$. Thus m must be less than 5.

Nevertheless, this strong condition can be modified if a quantile hedging approach is adopted, like for the Value-at-Risk (see Föllmer and Leukert (1999) for recent application of this notion in financial modelling): for example, an auxiliary floor can be chosen above the initial floor and the new condition is to guarantee that the portfolio value will be always above this new floor at a given probability $1-\varepsilon$. This gives the following relation for a period $[0, T']$:

$$P[C_t \geq 0, \forall t \in [0, T']] \geq \varepsilon$$

or, equivalently, if $M_{T'}$ denotes the maximum of the X_k for times t_k in $[0, T']$:

$$P\left[\forall t_k \in [0, T'], X_k \leq \frac{1}{m}\right] = P\left[M_{T'} \leq \frac{1}{m}\right] \geq 1 - \varepsilon.$$

Note that, since m is non negative, the condition $X_k \leq \frac{1}{m}$ is equivalent to $X_k\, 1_{0 \leq X_k} \leq \frac{1}{m}$.

Consider now that, on the given management period, the possible values of X_k are in the interval $[a, b]$ with probability one (in practice, we can take $b = 0, 1$ which corresponds to a maximal drop equal to 10%). The truncated values $X_{k+1}^{[a,b]}$ denotes the truncated jumps $(X_{k+1} 1_{a \leq X_{k+1} \leq b})$ and $(T_{k+1}^{[a,b]})_k$ is the corresponding sequence of arrival times. Define $M_{T'}^{[a,b]}$ as the maximum of the $X_k^{[a,b]}$ for times t_k in $[0, T']$. The quantile hedging condition leads to:

$$P\left[\forall t_k \in [0, T'], X_k^{(a,b]} \leq \frac{1}{m}]\right] = P\left[M_{T'}^{[a,b]} \leq \frac{1}{m}\right] \geq 1 - \varepsilon.$$

As it has been previously mentioned, the times $T_k^{[a,b]}$ are random variables and the range of their distributions is the set of times $(t_k)_k$. Suppose for example that the number of times t_k is sufficiently great so that we can consider that the set of t_k seems like an interval (continuous time at the limit). Assume also that the sequence of interarrival times $(T_{k+1}^{[a,b]} - T_k^{[a,b]})_k$ is an iid sequence and so is exponentially distributed with a parameter $\lambda^{[a,b]}$. This implies that the expectation of the number of variations with values in $[a, b]$, during the period $[0, T']$, is equal to $T' \lambda^{[a,b]}$. Then we can detail the quantile hedging condition. For this, introduce the function $F^{[a,b]^{-1}}$ defined as the inverse of the distribution function $F^{[a,b]}$ which is assumed to be strictly increasing. Then, we get (see Prigent (2001*b*)):

Proposition 2:

$$m \leq \frac{1}{F^{[a,b]^{-1}}\left(1 + \frac{\ln(1-\varepsilon)}{\lambda^{[a,b]}T'}\right)}.$$

This condition gives an upper limit on the multiple m which is obviously greater than the standard limit $\frac{1}{b}$ (which is given in proposition 1 with $b = d$ if the distribution is not truncated). It takes account of both the distribution of the variations and of the interarrival times. Note that if the intensity $\lambda^{[a,b]}$ increases, then this upper limit is decreasing. So, as intuition suggests, if the frequency $\lambda^{[a,b]}$ increases, then the multiple has to be reduced and if $\lambda^{[a,b]}$ goes to infinity, then the previous upper limit converges to the standard limit $\frac{1}{b}$.

Denote N the number of dates during the management period $[0, T']$. Suppose now that the true distribution $F^{[a,b]}$ is not known:

First case: assume that we have no prior information, so the X_k are not truncated and the arrival times are the t_k themselves.

Then the insurance condition on the multiple m can be still analysed by applying the extreme value theory. Indeed we know that there exist scale and location parameters ψ_N and μ_N such that $\frac{M_{T'} - \mu_N}{\psi_N}$ converges in distribution to one of the extreme value distributions H_ξ. Consequently, we can get the following approximation of the upper limit on the multiple m:

Proposition 3:

$$m \leq \frac{1}{\psi_N \, H_\xi^{-1}(1-\varepsilon) + \mu_N}.$$

Proof: from the condition $P\left[M_{T'} \leq \frac{1}{m}\right] \geq 1 - \varepsilon$, and by applying the extreme value theory to $M_{T'}$, we get $H_\xi\left(\frac{\frac{1}{m} - \mu_N}{\psi_N}\right) \geq 1 - \varepsilon$, which leads to the result.

Second case: When, due to anticipation, truncations are introduced, we can get also an upper bound.

Now, the times of prices variations are random variables, defined on the set of $(t_k)_k$. Denote by $\mathcal{N}_{T'}^{[a,b]}$ the random number of random variables X_k with values in the interval $[a, b]$ during the period $[0, T']$. Let also $T_k^{[a,b]}$ be the corresponding random arrival times. The quantile hedging condition becomes:

$$P[\forall t \in [0, T'], C_t \geq 0] = P\left[\forall T_k^{[a,b]} \leq T', X_k \leq \frac{1}{m}\right].$$

Then, by conditioning with respect to the number $\mathcal{N}_{T'}^{[a,b]}$, we obtain:

$$P[\forall T_k^{[a,b]} \leq T', X_k \leq \frac{1}{m}] = \sum_{k \leq N} P\left[M_k^{[a,b]} \leq \frac{1}{m};\ \mathcal{N}_{T'}^{[a,b]} = k\right]$$

$$= \sum_{k \leq N} P\left[M_k^{[a,b]} \leq \frac{1}{m};\ \mathcal{N}_{T'}^{[a,b]} = k\right] P[\mathcal{N}_{T'}^{[a,b]} = k]$$

$$= \sum_{k \leq N} P\left[M_k^{[a,b]} \leq \frac{1}{m}; T_k^{[a,b]} \leq T' < T_{k+1}^{[a,b]}\right] P[T_k^{[a,b]} \leq T' < T_{k+1}^{[a,b]}].$$

Introduce the function $L_{T'}^{[a,b]}$ defined by:

$$L_{T'}^{[a,b]}(x) = \sum_{k \leq N} P[M_k^{[a,b]} \leq x; T_k^{[a,b]} \leq T' < T_{k+1}^{[a,b]}]\, P[T_k^{[a,b]} \leq T' < T_{k+1}^{[a,b]}].$$

Since $F^{[a,b]}$ is assumed to be strictly increasing then $L_{T'}^{[a,b]}$ has also the same property. Consider its inverse $L_{T'}^{[a,b]^{-1}}$. We get immediately the following result:

Proposition 4:

The general quantile hedging condition $P[C_t \geq 0, \forall t \leq T'] \geq 1 - \varepsilon$ is equivalent to:

$$m \leq \frac{1}{L_{T'}^{[a,b]^{-1}}(1 - \varepsilon)}.$$

As it can be seen, this condition involves the joint conditional distributions of the marked point process $(T_{k+1}^{[a,b]}, X_{k+1}^{[a,b]})_k$. In the independent marking case, where $(T_k^{[a,b]})_k$ and $(X_k^{[a,b]})_k$ are independent, we get the following corollary:

Corollary 1:

$$L_{T'}^{[a,b]}(x) = \sum_{k \leq N} P[M_k^{[a,b]} \leq x]\, P[\mathcal{N}_{T'}^{[a,b]} = k],$$

and, for the special case of Independent marked Poisson process:

$$L_{T'}^{[a,b]}(x) = \sum_{k \leq N} P[M_k^{[a,b]} \leq x]\, \frac{(\lambda^{[a,b]} T')^k}{k!}\, e^{-\lambda^{[a,b]} T'}.$$

Remark: Estimation of the function $L_{T'}^{[a,b]}(x)$ in the independent case.

If T' is sufficiently great, the probabilities $P[\mathcal{N}_{T'}^{[a,b]} = k]$ are small, for small values of k. So, by applying the extreme value theory, we can get an approximation of each $P[M_k^{[a,b]} \leq x; T_k^{[a,b]} \leq T' < T_{k+1}^{[a,b]}]$. Thus, we get:

$$L_{T'}^{[a,b]}(x) \simeq \sum_{k \leq N} H_{\xi}^{[a,b]}\left(\frac{x - \mu_k^{[a,b]}}{\psi_k^{[a,b]}}\right) P[T_k^{[a,b]} \leq T' < T_{k+1}^{[a,b]}].$$

3.2. Empirical estimations

3.2.1. Estimations in the non truncated case

A. Estimations of the variations X_k

We examine the variations X_k, opposite of the arithmetic returns (4), of the S&P 500 during the period 01/1969 - 09/1997. We first consider the whole support of X_k.

Table 1

	S&P 500
Mean	−0.029331
Median	0.000000
Maximum	20.41387
Minimum	−9.099354
Std	0.895953
Skewness	1.515078
Kurtosis	44.18822
Jarque-Bera	532163

We refer to Longin (1996) for details about estimation procedures when dealing with extreme value theory. We begin with some usual descriptive statistics.

As expected, X_k is far from being normal distributed and has fat tailed. We find evidence for heteroskedasticity and autocorrelation in the series.

More precisely, the original series should be replaced by the innovation of an $AR(5)$ [5]. Nevertheless, the estimates on the raw data don't differ qualitatively from the ones on innovations [6]. As we need estimates on the raw data for the upper bound on the multiple, we conduct econometrics work on the original series.

We are now interested in the estimation of the tail parameter $\xi \in \mathbb{R}$ under the assumption that $X_1, \ldots, X_n$ are iid from $F \in MDA(H_\xi)$. We use Pickand's estimator.

Let $X_{k,n} < \ldots < X_{n,n}$ be an ordered sample of size k, Pickand's estimator is defined by:

$$\hat{\xi}^P_{k,n} = \frac{1}{\mathrm{Ln}(2)} \mathrm{Ln} \left[\frac{X_{k,n} - X_{2k,n}}{X_{2k,n} - X_{4k,n}} \right].$$

It can be shown that $\hat{\xi}^P_{k,n}$ is consistent provided $k \to \infty$, $\frac{k}{n} \to 0$.

Moreover, it is asymptotically normal:

$$\sqrt{k}\left(\hat{\xi}^P_{k,n} - \xi\right) \xrightarrow{d} \mathcal{N}(0, v(\xi))$$

where:

$$v(\xi) = \frac{\xi^2 (2^{2\xi+1} + 1)}{(2(2^\xi - 1)\,\mathrm{Ln}(2))^2}.$$

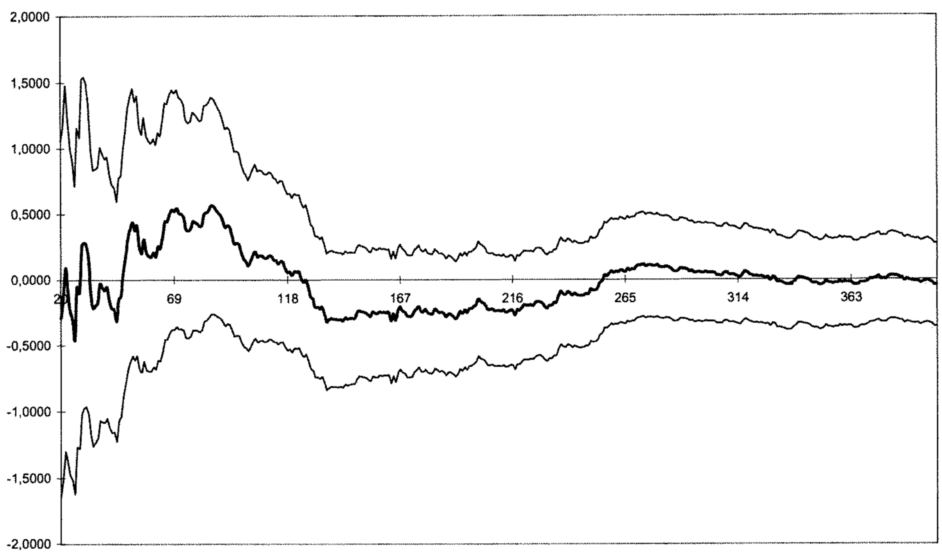

Figure 1: Pickand's estimate of the tail index

We can check on the above graphic (k on the x axis and $\hat{\xi}^P_{k,n}$ on the y axis), that the Pickand estimate of the tail index seems to stabilize on $\xi = 0$. This is in favor of a Gumbell distribution for the maximum domain of attraction of F.

We now turn to the statistical method which consists in estimating the GEV distribution over subperiods. We first plot the histogram of the maximum of $\boldsymbol{X}_k$ over 20 days.

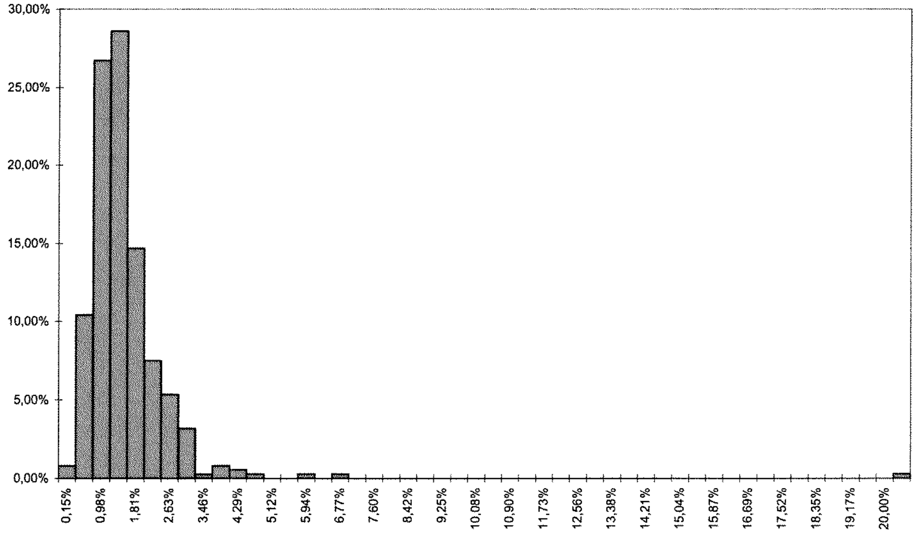

Figure 2: Histogram for max of 20-histories

Notice first that, on our sample, the maximal values of $\boldsymbol{X}_k$ are always positive. Knowing that we have to discriminate between the Weibull and the Gumbell distribution, the histogram confirms that the extremes of $\boldsymbol{X}_k$ seems to follow a Gumbell distribution.

We then consider the quantile plot, QQ-plot, over the same subperiods. Denote $M^{(l)}$ the maxima of $\boldsymbol{X}_k$ over the subperiods (or *blocks*) l $(l = 1, \ldots, p)$. Recall that θ is the length of each block l. The QQ-plot, on the ordered sample $M^{(p,p)} < \ldots < M^{(1,p)}$, is the graph of:

$$\left\{\left(M^{(l,p)}, F^{\leftarrow}\left(\frac{p-l+1}{p+1}\right)\right) : l = 1, \ldots, p\right\}.$$

Usually, one tests the empirical ordered sample against a Gumbell distribution where the inverse distribution at point x is given by $-\operatorname{Ln}(-\operatorname{Ln}(x))$. Convexity of the QQ-plot means that the data are generated by a Weibull distribution, linearity means that the limit distribution is a Gumbell one and concavity means that the limit distribution is a Frechet one.

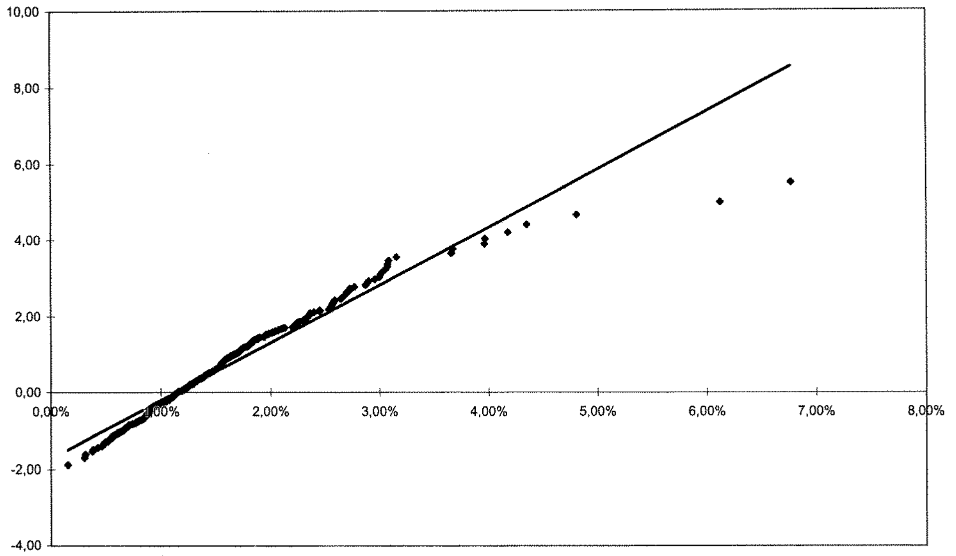

Figure 3: QQ-Plot for max of 20-histories without the maximal observations

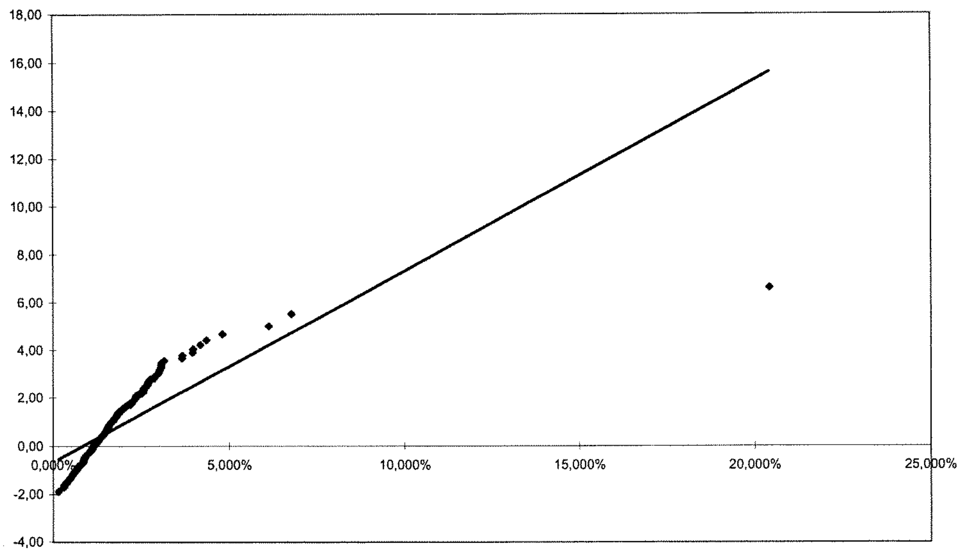

Figure 4: QQ-Plot for max of 20-histories

The curve on the first graph corresponds to the sample without the maximal value (it is the set of the sufficiently observed datas). It is roughly linear, at least up to 5% (note that the probability that the values X_k are smaller than 5% is almost equal to one). The curve on the second graph includes the maximal value. Clearly, it is not convex, so the Weibull distribution is excluded. As we only have the choice between the Gumbell and the Weibull distribution, we conclude in favor of the former.

Now, we consider the maximum likelihood estimation of GEV distribution of the non normalized series. The density is given by:

$$h_{\xi,\mu,\psi}(x) = \begin{cases} \dfrac{1}{\psi}\left(1+\xi\left(\dfrac{x-\mu}{\psi}\right)\right)^{\frac{-1}{\xi}-1} \exp\left(-\left(1+\xi\left(\dfrac{x-\mu}{\psi}\right)\right)^{\frac{-1}{\xi}}\right) & \text{if } 1+\xi\left(\dfrac{x-\mu}{\psi}\right) > 0 \text{ and } \xi \neq 0 \\ \dfrac{1}{\psi}\exp\left(-\dfrac{x-\mu}{\psi} - \exp\left(-\left(\dfrac{x-\mu}{\psi}\right)\right)\right) & \text{if } \xi = 0. \end{cases}$$

The likelihood function to maximize is given by:

$$L(\xi, \mu, \psi) = \prod_{l=1}^{p} h_{\xi,\mu,\psi}(M^{(l)})\, 1_{1+\xi\left(\frac{M^{(l)}-\mu}{\psi}\right)>0}.$$

The estimation results for a Gumbell distribution and for various length of the subperiod θ are reported in the following table (7):

Table 2

Length of the selection period	Scale parameters μ_θ	Location parameters ψ_θ
$\theta = 20$	1.193427	0.579517
	(0.031254)	(0.024154)
$\theta = 60$	1.680853	0.703115
	(0.065222)	(0.053400)
$\theta = 120$	1.993703	0.899447
	(0.117362)	(0.098554)
$\theta = 240$	2.474917	1.135238
	(0.20846)	(0.208460)

B. Estimations of the upper bound

We are now able to give an estimation of the upper bound on the multiple. From proposition 3, the upper bound is given by $\dfrac{1}{\psi_N\, H_\xi^{-1}(1-\varepsilon)+\mu_N}$, where N denotes the number of transaction dates during the management period $[0, T]$. The previous estimation has shown that we must take H_ξ equal to the

Table 3

Number of transaction dates	$\varepsilon = 5\%$	$\varepsilon = 1\%$	$\varepsilon = 0,1\%$
$N = 60$	26,53	20,34	15,3
$N = 120$	21,44	16,31	12,19
$N = 240$	17,10	12,99	9,69

Gumbell distribution. Thus, the estimation of this upper bound is equal to $\dfrac{1}{\psi_N[-\text{Ln}(-\text{Ln}(1-\varepsilon))] + \mu_N}$.

As expected, the upper bound on the multiple is decreasing with respect to the risk aversion of the bank, $1 - \varepsilon$, and to the number of transaction dates, N, during the management period $[0, T']$.

If bank risk aversion is very high, say $\varepsilon = 0,1\%$, the upper bound on the multiple is in line with the usual value of the multiple used by practitioners (between 7 and 10). As soon as bank risk tolerance is higher, $\varepsilon = 1\%$, the multiple is above its usual value: 12,99.

3.2.2. Estimations in the truncated case

A. Estimations of the interarrival times distribution

When dealing with the sequence of interarrival times $T_{k+1}^{[a,b]} - T_k^{[a,b]}$ along the whole period of observations, it is possible to examine the influence of the interval $[a; b]$ on the frequency λ of the daily variations in these intervals (the estimation of λ is deduced from the average of the $T_{k+1}^{[a,b]} - T_k^{[a,b]}$). This is illustrated by the following table:

Table 4

$[a;b]$	λ	$[a;b]$	λ	$[a;b]$	λ
$[-1\%;0]$	0,40045	$[-2\%;0]$	0,48044	$[-3\%;0]$	0,49232
$[-1,25\%;0]$	0,43424	$[-2,25\%;0]$	0,48458	$[-3,25\%;0]$	0,49326
$[-1,5\%;0]$	0,45614	$[-2,5\%;0]$	0,48805	$[-3,5\%;0]$	0,49326
$[-1,75\%;0]$	0,47256	$[-2,75\%;0]$	0,49072	$[-3,75\%;0]$	0,49366

We observe that the frequency converges to approximately 0.5 as $[a; b]$ increases: this is due to the fact that daily returns have almost the same probability to be positive than to be negative.

For each period $[0, T']$, the distribution of the number of variations in the interval $[a, b]$ is estimated. This gives the values of the probabilities $P[\mathcal{N}_{T'}^{[a,b]} = k]$. The next table indicates the kind of distribution that we can get. We choose $[a, b] = [0,10\%]$, $T' = 1$ year. N_v is the number of variations in this interval. *Nd* is the number of days. *f* is the corresponding frequency. The period of observation is 01/1971-09/1997.

Table 5

Nv	Nd	f	Nv	Nd	f	Nv	Nd	f
124	261	47.51%	117	262	44.66%	109	260	41.92%
123	260	47.31%	144	261	55.17%	125	261	47.89%
144	261	57.17%	146	261	55.94%	136	261	52.11%
152	261	58.24%	123	260	47.31%	132	262	50.38%
124	261	47.51%	147	261	56.32%	131	261	50.19%
126	262	48.09%	126	262	48.09%	127	260	48.85%
142	260	54.62%	122	261	46.74%	104	260	40.00%
130	260	50.00%	111	261	42.53%	123	262	46.95%
121	261	46.36%	121	261	46.36%	83	184	45.11%

B. Estimations of the upper bound

Applying the previous results, we can provide an approximation of the upper bound for the multiple if we assume that the variations are conditionally independent from the times of variations. Consider a period $[0, T']$ which corresponds to one year. From table 4, we deduce that the range of the distribution of $\mathcal{N}_{T'}^{[0,10\%]}$ is [100;150] and we get estimates for $P[\mathcal{N}_{T'}^{[0,10\%]} = k]$. Then, we take the values of the norming constants $\mu_N^{[0,10\%]}$ and $\psi_N^{[0,10\%]}$ for N in [100;150], which are estimated by a similar maximum likelihood method as in the non truncated case. Finally, using the approximation given by the extreme value theory, we deduce:

$$L_{T'}^{[0,10\%]}(x) \simeq \sum_{100 \leq k \leq 150} \Lambda\left(\frac{x - \mu_k^{[0,10\%]}}{\psi_k^{[0,10\%]}}\right) P[\mathcal{N}_{T'}^{[0,10\%]}] = k].$$

Now, if we want to hedge at the level $1 - \varepsilon$, by applying the relation of proposition 4, we obtain the following table:

Table 6

Length of the management period	$\varepsilon = 5\ \%$	$\varepsilon = 1\ \%$	$\varepsilon = 0,1\ \%$
$T = 240$	23, 16	17, 92	13, 57

We can check that this values are all greater than 10 and are also higher than the values obtained on the whole support.

So, a portfolio manager who doesn't think that a daily drop greater than 10% could occur before he has a chance to trade, can use higher multiple value than in the previous case. Notice that the upper bound at a 1% risk level becomes 17,92 (to compare with 12,99), and even if bank risk aversion is very high ($\varepsilon = 0,1\%$), the upper bound becomes 13,57 instead of 9,69.

4. Conclusion

As it can be seen, it is possible to choose higher multiples for the CPPI method if quantile hedging is used. The upper bounds can be calculated for each level of probability, according to the distributions of the marked point process which indicates the variations of the underlying asset and its times of variations. The extreme value theory allows to approximate these distributions to provide a numerical upper bound. We have illustrated this result on S&P 500 data. The difference with the standard multiple is significant, especially if we consider that the larger historical daily market decrease is unlikely to appear during the management period. This result has been established using some independence assumptions.

The statistical study of general dependent marking (in particular, the introduction of exogeneous processes such as economic variables) will be the purpose of further research.

Notes

(1) Customers indirectly bear part of the risk in that they abandon part of the expected return when they invest in an insured portfolio.

(2) For example, portfolio managers cannot actually rebalance portfolios in continuous time. Moreover, problems of asset liquidity may occur, especially during stock markets crashes (see Longin (1997)).

(3) For the basic definitions and properties about marked point processes, we refer for example to Bremaud (1981), Last and Brandt (1995) for marked point processes on the real line and Jacod (1977) for more general multivariate point processes.

(4) More precisely, we conduct empirical investigations on $100 . X_k$.

(5) We have used the Akaike's Information Criterion (AIC) to choose the appropriate model. In fact, an $AR(6)$ has emerged but, due to the well known problem of overparametrization of the AIC and to the fact that 5 days are a week of exchange, we select an $AR(5)$ model.

(6) The same kind of results can be found in Jondeau and Rockinger (1999).

(7) Recall that we are working on $100 . X_k$. Standard errors of parameters estimates are given in parentheses. For $\theta = 240$, we are well aware that the estimation precision is not completely sufficient. Nevertheless, it has seemed to us interesting to report the corresponding results.

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